Formal derivation of seawater intrusion models

Mustapha Jazar, Régis Monneau

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February 25, 2012

Abstract

In this paper, we consider the flow of fresh and saltwater in a saturated porous medium in order to describe the seawater intrusion. Starting from a formulation with constant densities respectively of fresh and of saltwater, whose velocity is proportional to the gradient of pressure (Darcy's law), we consider the formal asymptotic shallow water limit as the ratio between the thickness and the horizontal length of the porous medium tends to zero. In this limit, we derive the Dupuit-Forchheimer condition and as a consequence reduced models of Boussinesq type both in the cases of unconfined and confined aquifers.

MSC: 35R35, 35B40

Keywords: seawater intrusion, formal asymptotics, porous medium, groundwater flow, Dupuit-Forchheimer, saltwater and freshwater interface, Ghyben-Herzberg relation, confined and unconfined aquifer, shallow water, Boussinesq equation.

1 Introduction

We are interested in the modeling of seawater intrusion in coastal regions. On the one hand coastal aquifers contain freshwater and on the other hand saltwater from the sea can enter in the ground and replace the freshwater. This phenomenon can be especially important in coastal regions with intensive extraction of freshwater in wells. We refer to [7] for a general overview on seawater intrusion models.

Our main goal is to derive formally simplified (2D) models describing the evolution of the interfaces freshwater/saltwater and freshwater/ dry soil, from common (3D) models of hydrology based on the Darcy's law.

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1.1 Setting of the problem

We consider two simple situations: the case of an unconfined aquifer and the case of a confined aquifer.

1.1.1 The unconfined aquifer

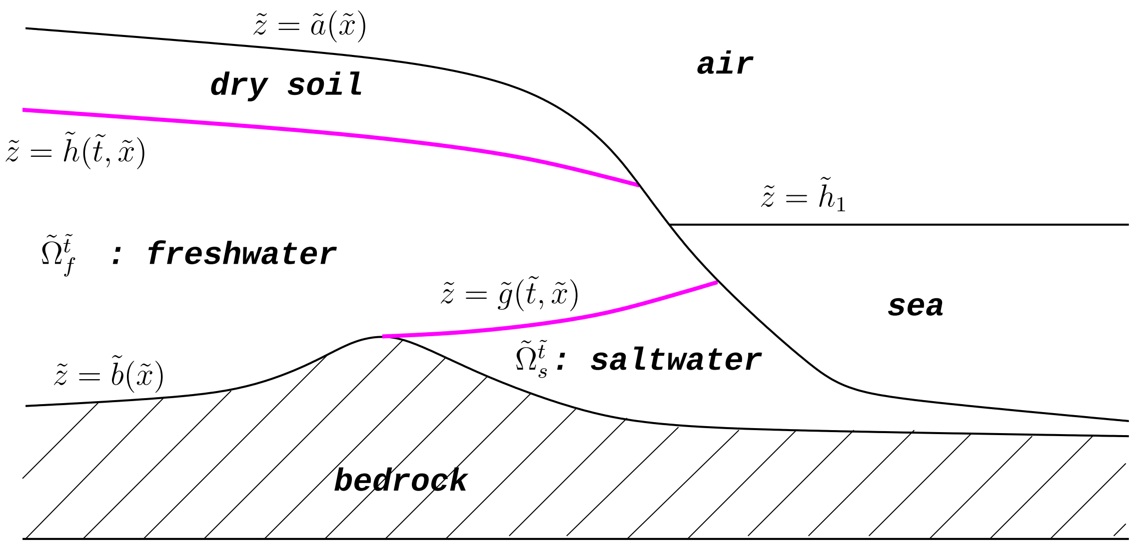

Figure 1: Unconfined aquifer

The geometry
We consider coordinates $(\tilde x, \tilde z) \in \mathbb{R}^N \times \mathbb{R}$ of the space with $\tilde x$ for the horizontal coordinate and $\tilde z$ for the vertical coordinate. In physical application, we have $N = 1$ or $N = 2$. We assume (see Figure 1) that the surface of the soil is described by the level $\tilde z = \tilde a(\tilde x)$, while the interface with the impermeable bedrock is described by the level $\tilde z = \tilde b(\tilde x)$, satisfying $\tilde b \le \tilde a$. We assume that in the porous medium, the interface between the freshwater and the dry soil can be written

$$\Gamma_{\tilde h}^{\tilde t} = \left\{(\tilde x, \tilde z) \in \mathbb{R}^N \times \mathbb{R}, \quad \tilde z = \tilde h(\tilde t, \tilde x)\right\}$$

and the interface between the saltwater and the freshwater (which are assumed to be immiscible) can be written

$$\Gamma_{\tilde g}^{\tilde t} = \left\{(\tilde x, \tilde z) \in \mathbb{R}^N \times \mathbb{R}, \quad \tilde z = \tilde g(\tilde t, \tilde x)\right\}$$

and that we have the following constraint

$$\tilde b \le \tilde g \le \tilde h \le \tilde a \quad \text{on} \quad \mathbb{R}^N. \tag{1.1}$$

We assume that all the functions $\tilde b, \tilde g, \tilde h, \tilde a$ are smooth enough. We define the open set of porous medium as

$$\tilde\Omega = \left\{(\tilde x, \tilde z) \in \mathbb{R}^N \times \mathbb{R}, \quad \tilde z < \tilde a(\tilde x)\right\}$$

the open set of freshwater

$$\tilde\Omega_f^{\tilde t} = \left\{(\tilde x, \tilde z) \in \mathbb{R}^N \times \mathbb{R}, \quad \tilde g(\tilde t, \tilde x) < \tilde z < \tilde h(\tilde t, \tilde x)\right\}$$

and the open set of saltwater in the porous medium

$$\tilde{\Omega}_s^{\tilde{t}} = \left\{ (\tilde{x}, \tilde{z}) \in \mathbb{R}^N \times \mathbb{R}, \quad \tilde{b}(\tilde{x}) < \tilde{z} < \tilde{g}(\tilde{t}, \tilde{x}) \right\}.$$

The PDEs

We set $\alpha = f$ for the freshwater and $\alpha = s$ for the saltwater. We define the density field of the fluid α as

$$\tilde{\rho}_\alpha(\tilde{t}, \tilde{x}, \tilde{z}) = \begin{cases} \rho_\alpha^0 & \text{if} \quad (\tilde{x}, \tilde{z}) \in \tilde{\Omega}_\alpha^{\tilde{t}} \\ 0 & \text{otherwise} \end{cases}$$

where ρ_α^0 is the volumic mass of the fluid α (assumed to be a constant with $0 < \rho_f^0 < \rho_s^0$). We also set the specific weight $\gamma_\alpha = \rho_\alpha^0 g^0$ with g^0 the standard gravity constant. We assume that $\tilde{\rho}_\alpha$ solves the following equations

$$(1.2) \qquad \begin{cases} \left.\begin{array}{l} \tilde{\phi}_\alpha(\tilde{x}, \tilde{z}) \dfrac{\partial \tilde{\rho}_\alpha}{\partial \tilde{t}} + \widetilde{\operatorname{div}}\, (\tilde{\rho}_\alpha \tilde{v}_\alpha) = 0 \quad \text{in} \quad \mathcal{D}'\left((0, +\infty) \times \tilde{\Omega}\right) \\ \tilde{v}_\alpha = -\tilde{\kappa}_\alpha(\tilde{x}, \tilde{z})\, \widetilde{\nabla}(\tilde{p} + \gamma_\alpha \tilde{z}) \quad \text{in} \quad \tilde{\Omega}_\alpha^{\tilde{t}} \end{array}\right| \quad \text{for} \quad \alpha = f, s \\ \tilde{p} \quad \text{is continuous on} \quad \Gamma_{\tilde{g}}^{\tilde{t}} \\ \tilde{v}_f \cdot \tilde{n} \geq 0 \quad \text{on} \quad (\partial \tilde{\Omega}) \cap (\partial \tilde{\Omega}_f^{\tilde{t}}) \\ \tilde{v}_s \cdot \tilde{n} \geq 0 \quad \text{on} \quad (\partial \tilde{\Omega}) \cap (\partial \tilde{\Omega}_s^{\tilde{t}}) \cap \left\{ \tilde{z} > \tilde{h}_1 \right\} \end{cases}$$

where $\tilde{v}_\alpha$ is the Darcy's flux and $\tilde{v}_\alpha / \tilde{\phi}_\alpha$ is the velocity vector field of the fluid α. Here $\tilde{p}$ is the pressure assumed to be defined on $\overline{\tilde{\Omega}_f^{\tilde{t}} \cup \tilde{\Omega}_s^{\tilde{t}}}$, and $\widetilde{\operatorname{div}}$ and $\widetilde{\nabla}$ are respectively the divergence and the gradient taken with respect to the coordinates $(\tilde{x}, \tilde{z})$. Moreover $\tilde{\kappa}_\alpha(\tilde{x}, \tilde{z}) \in \mathbb{R}_{\text{sym}}^{(N+1)\times(N+1)}$ is a given symmetric matrix which is positive definite and $0 < \tilde{\phi}_\alpha(\tilde{x}, \tilde{z}) \leq 1$ is the effective porosity of the porous medium, where, in order to simplify for a fully saturated medium, we assume that the water content is equal to the porosity. Notice that this effective porosity $\tilde{\phi}_\alpha$ should be independent on the fluid α, but for sake of generality we allow here such a dependence. The expression defining the flux $\tilde{v}_\alpha$ follows from Darcy's law (where $\tilde{\kappa}_\alpha = \dfrac{k_\alpha}{\mu_\alpha}$ with μ_α is the dynamic viscosity and k_α is the intrinsec permeability tensor of the porous medium). The fourth condition of (1.2) involves the outward unit normal $\tilde{n}$ to $\tilde{\Omega}$ and means that the flux of fresh water can only go out of the soil (in the absence of sources). Similarly the fifth condition of (1.2) means that the flux of salt water can only go out of the soil, if the level is above the sea level $\tilde{z} = \tilde{h}_1$. Obviously, we do not really expect such a situation in practice, but if for some reasons it would happen, then we assume the fifth condition above.

We also assume the following boundary condition

(1.3)
$$\begin{cases} \tilde p(\tilde t,\tilde x,\tilde z) = \begin{cases} 0 & \text{if} \quad \tilde z = \tilde h(\tilde t,\tilde x) < \tilde a(\tilde x) \quad \text{or} \quad \tilde z = \tilde h(\tilde t,\tilde x) = \tilde a(\tilde x) \geq \tilde h_1 \\ \gamma_s(\tilde h_1 - \tilde z) & \text{if} \quad \tilde z = \tilde h(\tilde t,\tilde x) = \tilde a(\tilde x) < \tilde h_1 \end{cases} \\ \tilde h(\tilde t,\tilde x) = \tilde a(\tilde x) \quad \text{if} \quad \tilde a(\tilde x) < \tilde h_1 \end{cases}$$

The first condition of (1.3) follows from the fact that we assume the atmospheric pressure to be constant and normalized to zero and that the seawater is assumed to be at the hydrostatic equilibrium. We recall that the surface of the sea is assumed to be at the altitude $\tilde h_1$. When the free surface $\left\{\tilde z = \tilde h(\tilde t,\tilde x)\right\}$ has no contact with the sea, then its pressure is assumed to be equal to the atmospheric pressure zero. The last two lines of (1.3) mean that we assume that the part $\left\{\tilde a < \tilde h_1\right\}$ is under the seawater.

Notice that the evolution equations of the free boundary $\Gamma^{\tilde t}_{\tilde h}$ and $\Gamma^{\tilde t}_{\tilde g}$ follow from the first line of (1.2) in the sense of distributions.

1.1.2 The confined aquifer

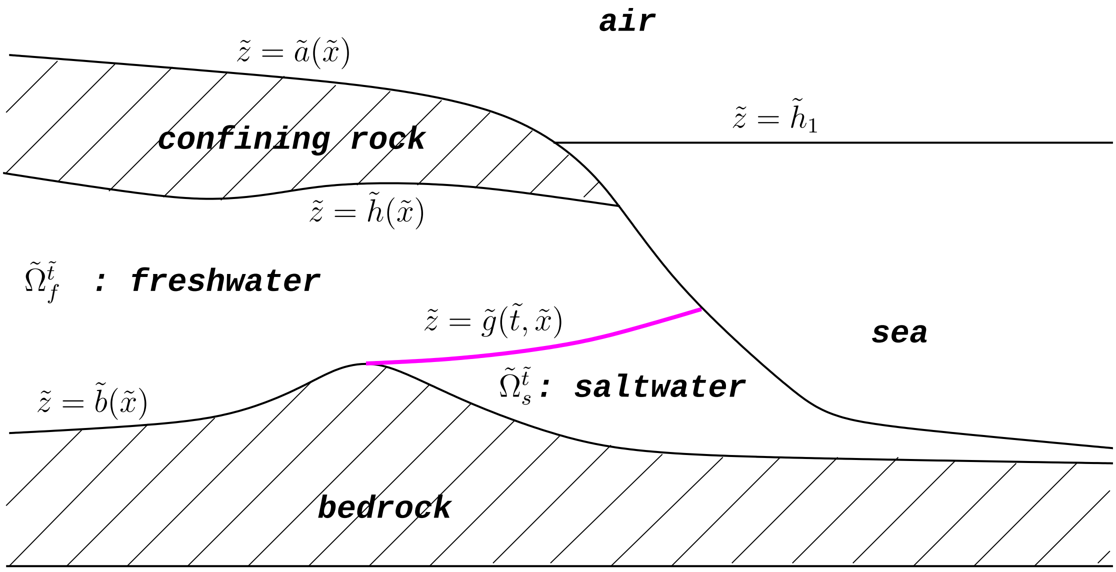

Figure 2: Confined aquifer

The situation of the confined aquifer is similar to the unconfined aquifer (see Figure 2). The main difference is that the function $\tilde h(\tilde x)$ is now a given function describing the shape of the upper confining aquifer and then is independent on time $\tilde t$. Therefore, we can write $\Gamma^{\tilde t}_{\tilde h} =: \Gamma_{\tilde h}$ as a time independent interface. Then equations (1.2) are still satisfied, and the boundary condition (1.3) is replaced by

(1.4) $$\tilde p(\tilde t,\tilde x,\tilde z) = \gamma_s \max(0,\tilde h_1 - \tilde z) > 0 \quad \text{for} \quad \tilde z = \tilde a(\tilde x) \quad \text{and} \quad \tilde x \in \mathbb{R}^N \backslash \omega$$

with the following open set

$$\omega = \left\{\tilde x \in \mathbb{R}^N, \quad \tilde h(\tilde x) < \tilde a(\tilde x)\right\}$$

which can be seen as the horizontal projection of the region where the fluid is fully confined. Notice that in order to simplify the presentation, we assume that the pressure in (1.4) is positive, which means that the unconfined part of the soil is under the sea (like on Figure 2).

1.2 Main result

We assume the existence of a small parameter $\varepsilon > 0$ such that the data of the problem satisfy

$$(1.5)\qquad \begin{cases} \tilde{x} = x \\ \tilde{z} = \varepsilon z \\ \tilde{t} = t/\varepsilon \\ \tilde{a}(\tilde{x}) = \varepsilon a(x) \\ \tilde{b}(\tilde{x}) = \varepsilon b(x) \\ \tilde{h}_1 = \varepsilon h_1 \\ \tilde{\phi}_\alpha(\tilde{x},\tilde{z}) = \phi_\alpha(x,z) \\ \tilde{\kappa}_\alpha(\tilde{x},\tilde{z}) = \kappa_\alpha(x,z) = \begin{pmatrix} \kappa_\alpha^{xx}(x,z) & \kappa_\alpha^{xz}(x,z) \\ & \\ \kappa_\alpha^{zx}(x,z) & \kappa_\alpha^{zz}(x,z) \end{pmatrix}. \end{cases}$$

The parameter ε can be seen as the ratio between the thickness of the soil (vertical dimension) and the horizontal length of the soil. Then we also rescale the functions describing the free boundaries of the problem as follows

$$(1.6)\qquad \begin{cases} \tilde{h}(\tilde{t},\tilde{x}) = \varepsilon h^\varepsilon(t,x) \quad \text{(with} \quad h^\varepsilon(t,x) = h(x) \quad \text{given in the confined case)} \\ \tilde{g}(\tilde{t},\tilde{x}) = \varepsilon g^\varepsilon(t,x). \end{cases}$$

We define

$$\mathcal{K}_\alpha(x,z) = \gamma_s \int_0^z d\bar{z}\ \bar{\kappa}_\alpha^{xx}(x,\bar{z}) \quad \text{with} \quad \bar{\kappa}_\alpha^{xx}(x,z) = \kappa_\alpha^{xx}(x,z) - (\kappa_\alpha^{zz}(x,z))^{-1}\kappa_\alpha^{xz}(x,z)\kappa_\alpha^{zx}(x,z)$$

and

$$\Phi_\alpha(x,z) = \int_0^z d\bar{z}\ \phi_\alpha(x,\bar{z})$$

and set

$$\varepsilon_0 = \frac{\gamma_s - \gamma_f}{\gamma_s} \in (0,1).$$

Unconfined aquifer

We have
(1.7)

$$\begin{cases} b \le g \le h \le a & \text{on} \quad [0,+\infty) \times \mathbb{R}^N \\ (\Phi_f(\cdot\,,h) - \Phi_f(\cdot\,,g))_t = \mathrm{div}_x \ \Big([\mathcal{K}_f(x,z)]_{z=g}^{z=h} \, \nabla_x (p + (1-\varepsilon_0)h) \Big) & \text{on} \quad \{h < a\} \\ (\Phi_f(\cdot\,,h) - \Phi_f(\cdot\,,g))_t \le \mathrm{div}_x \ \Big([\mathcal{K}_f(x,z)]_{z=g}^{z=h} \, \nabla_x (p + (1-\varepsilon_0)h) \Big) & \text{on} \quad [0,+\infty) \times \mathbb{R}^N \\ (\Phi_s(\cdot\,,g))_t = \mathrm{div}_x \ \big([\mathcal{K}_s(x,z)]_{z=b}^{z=g} \, \nabla_x (p + (1-\varepsilon_0)h + \varepsilon_0 g) \big) & \text{on} \quad \{g < a\} \\ (\Phi_s(\cdot\,,g))_t \le \mathrm{div}_x \ \big([\mathcal{K}_s(x,z)]_{z=b}^{z=g} \, \nabla_x (p + (1-\varepsilon_0)h + \varepsilon_0 g) \big) & \text{on} \quad [0,+\infty) \times \mathbb{R}^N \end{cases}$$

with

$$[\mathcal{K}_\alpha(x,z)]_{z=z_1}^{z=z_2} = \mathcal{K}_\alpha(x,z_2) - \mathcal{K}_\alpha(x,z_1).$$

and

$$(1.8) \qquad \begin{cases} h = a \quad \text{on} \quad \{a < h_1\} \\ p(t,x) = p_0(t,x) := \begin{cases} h_1 - a(x) & \text{if} \quad h = a < h_1 \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

Confined aquifer

Then we have (1.7) with

$$(1.9) \qquad \begin{cases} h_t = 0 \\ p(t,x) = p_0(x) := \max(0, h_1 - a(x)) > 0 \qquad \text{on} \quad \{h = a\} \end{cases}$$

Setting

$$\omega = \left\{ x \in \mathbb{R}^N, \quad h(x) < a(x) \right\}.$$

then p is in particular solution of
(1.10)

$$\begin{cases} (\Phi_s(\cdot\,,g) - \Phi_f(\cdot\,,g))_t \\ = \mathrm{div}_x \ \Big(\Big([\mathcal{K}_s(x,z)]_{z=b}^{z=g} + [\mathcal{K}_f(x,z)]_{z=g}^{z=h} \Big) \nabla_x (p + (1-\varepsilon_0)h) + \varepsilon_0 \, [\mathcal{K}_s(x,z)]_{z=b}^{z=g} \, \nabla_x g \Big) & \text{on} \quad \omega \\ p(t,x) = p_0(x) := \max(0, h_1 - a(x)) > 0 & \text{on} \quad \partial\omega. \end{cases}$$

Remark 1.1 *Notice that if $\phi_f = \phi_s$ (as it is expected in the physical problem), then we get zero on the left hand side of the first equation of (1.10) which becomes a stationary equation.*

We also introduce the following additional non-degeneracy condition:

$$(1.11) \qquad -\nu^T \cdot [\mathcal{K}_f(x,z)]_{z=g}^{z=a} \cdot \nabla_x a > 0 \quad \text{if} \quad g < a \quad \text{for} \quad x \in \partial\omega$$

where ν is the outward unit normal to ω.

Theorem 1.2 (Formal convergence and properties of the limit model)
a) (General convergence)
Under the previous assumptions, we have formally $(h^\varepsilon, g^\varepsilon) \to (h, g)$ *as* $\varepsilon \to 0$*, in the following cases:*
i) (unconfined case): *We assume (1.1)-(1.2)-(1.3). Then* (h, g) *solves (1.7)-(1.8).*
ii) (confined case): *We assume (1.1)-(1.2),(1.4). Then* (h, g) *solves (1.7),(1.9), and* p *solves in particular (1.10).*
b) (Further properties of the limit models)
i) (unconfined case): *We assume that* (h, g) *solves (1.7)-(1.8).*

- **i.1) (stationary case):**
  If $h_t = g_t = 0$*, then we have formally*

$$g = a = h \quad \text{on} \quad \{a < h_1\} \tag{1.12}$$

  if such relation holds at infinity.

- **i.2) (evolution case):**
  If (1.12) holds at time $t = 0$*, then it formally holds for all time, if for all time it holds at infinity in space.*

ii) (confined case): *We assume that* (h, g) *solves (1.7),(1.9).*
We assume moreover that the condition (1.11) is satisfied.

- **ii.1) (stationary case):**
  If $h_t = g_t = 0$*, then we have formally*

$$g = a \quad \text{on} \quad \{h = a\} \tag{1.13}$$

  if such relation holds at infinity.

- **ii.2) (evolution case):**
  If (1.13) holds at time $t = 0$*, then it formally holds for all time, if for all time it holds at infinity in space.*

Remark 1.3 *We do not know natural conditions to insure for all time for the unconfined model that* $g \le h_1 \le h$ *is true in* $\{a \ge h_1\}$*.*

Remark 1.4 *Notice that for the stationary limit model (1.7)-(1.8) (resp. (1.7),(1.9)), we always have* $g = a = h$ *on* $\{a < h_1\}$ *(resp. on* $\mathbb{R}^N \backslash \omega$*). This shows (at least formally) in the limit* $\varepsilon \to 0$*, that the region* $\left\{\tilde{g} < \tilde{a} < \tilde{h}_1\right\}$ *(resp.* $\{\tilde{g} < \tilde{a}\} \cap (\mathbb{R}^N \backslash \omega)$*) shrinks and disappears. The reader may have a look to Figures 1 and 2.*

Remark 1.5 *The coefficients* $[\mathcal{K}_f(x,z)]_{z=g}^{z=h}$ *and* $[\mathcal{K}_s(x,z)]_{z=b}^{z=g}$ *are sometimes called the transmitivity coefficients. Notice that in the case* $\mathcal{K}_\alpha(x,z) = zId$*,* $\phi_\alpha(x,z) = 1$ *and* $b = 0$*, equations (1.7) reduces to the Boussinesq equation (see equation (2) page 14 in [10]) either for* $g = b$ *(no saltwater) or for* $h = g$ *(no freshwater).*

Remark 1.6 *Notice that the inequality in the third line of (1.7) is a consequence of the outward velocity condition given in the fourth line of (1.2) (and similarly the fifth line of (1.7) is a consequence of the fifth line of (1.2)). Notice also that inequality (1.11) holds, if* $\mathcal{K}_f$ *is proportional to the identity and if on* $\partial\omega$*, the vector field* $\nabla_x a$ *points inwards* ω*.*

Remark 1.7 *Sources (or wells) of freshwater can be added on the right hand side of both the second and the third line of (1.7). For the confined case, this will modify consequently the equation (1.10) for the pressure.*

Theorem 1.8 (Sufficient condition for Ghyben-Herzberg relation)
We still work under the previous assumptions, as in Theorem 1.2.
i) (unconfined case)
Assume that $g \leq h_1$. If (h, g) solves (1.7)-(1.8) with $g_t = h_t = 0$, then the following Ghyben-Herzberg relation

$$p + (1 - \varepsilon_0)h + \varepsilon_0 g = h_1 \quad \textit{with} \quad p = 0 \tag{1.14}$$

holds on each connected component of $\{b < g < a\}$ whose boundary intersects $\{g = a\}$.
ii) (confined case)
If (h, g) solves (1.7), (1.9) with $g_t = h_t = 0$, then the following Ghyben-Herzberg relation

$$p + (1 - \varepsilon_0)h + \varepsilon_0 g = h_1 \tag{1.15}$$

holds on each connected component of $\{b < g < a\}$ whose boundary intersects $\{g = a\}$.

Remark 1.9 *Notice that there is no reason for the Ghyben-Herzberg condition to hold in the evolution case.*

Remark 1.10 (Stationary free boundary problem)
Notice that using the Ghyben-Herzberg condition, we see that both in the unconfined and the confined case, any solution (h, g) of the following free boundary system is also a stationary solution of (1.7):

$$\left\{\begin{array}{lll} b \leq h \leq a & \textit{on} & [0, +\infty) \times \mathbb{R}^N \\[2ex] g = \max\left(b,\ \dfrac{h_1 - p - (1 - \varepsilon_0)h}{\varepsilon_0}\right) & \textit{on} & \{g < a\} \\[2ex] 0 = div_x \ \left([\mathcal{K}_f(x, z)]_{z=g}^{z=h} \nabla_x (p + (1 - \varepsilon_0)h)\right) & \textit{on} & \{h < a\} \\[2ex] 0 \leq div_x \ \left([\mathcal{K}_f(x, z)]_{z=g}^{z=h} \nabla_x (p + (1 - \varepsilon_0)h)\right) & \textit{on} & [0, +\infty) \times \mathbb{R}^N \end{array}\right. \tag{1.16}$$

where p is given by (1.8) and we look for the solution h in the unconfined case, and h is given by (1.9) and we look for p in the confinded case.
Here the free boundary is $\partial \{g > b\}$.

Remark 1.11 (Simple reduced system)
Let us set $\bar{h} = h - g \geq 0$ and $\bar{g} = g - b \geq 0$. Then if $\mathcal{K}_\alpha(x, z) = zI$, $\phi_\alpha = 1$ and $b = 0$, we see that the system (1.7) has a particularly simple form with $\varepsilon_0 \in (0, 1)$, in the region $\{\bar{h} + \bar{g} < a - b\}$:
a) evolution case
a.i) unconfined case

$$\left\{\begin{array}{l} \bar{h}_t = div_x \ (\bar{h} \nabla_x \{p + (1 - \varepsilon_0)(\bar{h} + \bar{g})\}) \\[2ex] \bar{g}_t = div_x \ (\bar{g} \nabla_x \{p + (1 - \varepsilon_0)\bar{h} + \bar{g}\}) \end{array}\right. \quad \Bigg| \quad \textit{with} \quad p = 0 \tag{1.17}$$

and in the
a.ii) confined case

$$\text{(1.18)} \qquad \textit{system (1.17) with the pressure } p \textit{ determined by } (\bar{h}+\bar{g})_t = 0$$

b) stationary case (a free boundary problem)
If $g = \max\big(b, h_1 - p - (1-\varepsilon_0)\bar{h}\big)$ *on* $\{g < a\}$ *(which satisfies the Ghyben-Herzberg condition), we get the following free boundary problems:*
b.i) unconfined case

$$\text{(1.19)} \qquad 0 = div_x\ \left(\bar{h}\nabla_x \left\{\max\left(\bar{h}, \varepsilon_0\bar{h} + h_1\right)\right\}\right)$$

b.ii) confined case

$$\text{(1.20)} \qquad \begin{cases} 0 = \Delta w \quad \textit{with} \quad w = \varepsilon_0^{-1}(p - (h_1 - h_0)) + \dfrac{\bar{g}^2}{2h_0} \\ \\ \bar{g} - \dfrac{\bar{g}^2}{2h_0} + w = h_0 \quad \textit{in} \quad \{\bar{g} > 0\} = \{w < h_0\} \end{cases}$$

when h *is constant and satisfies* $h = h_0 \in (0, h_1)$ *in* $\{h < a\}$ *and where expressions (1.20) are given in Proposition 4.5.*

1.3 Brief review of some retated literature

The literature on the subject is huge. We only give here some indications on the literature.

For general models of groundwater flows, we refer the reader to [7, 6, 11], where basically we can find two kind of models: sharp interface models (that we consider in the present paper) and models with variable concentration of salt. For mathematical analysis, see [12] for models with variable densities or [19] with diphasic models with capillarity pressure. For more historical notes on the origin of the models, see [14, 20, 27, 15].

Sharp interface models in the stationary regime have been studied mathematically, see for instance [13] for one phase problems and [3, 9, 2] for two phase problems.

For 2D models describing interfaces, we refer the reader to [10] where Boussinesq derived the porous medium equation under certain assumptions. See the recent book of Vàzquez [35] for the mathematical study of this equation. Starting from sharp interface models, certain derivation of 2D models under certain assumptions are derived in hydrology in [5, 8, 18, 4]. See also [1, 24] for some applications. Different models are derived in [31, 29, 32] in the framework of variable concentration of salt.

Notice that the method to deduce 2D models from 3D models is similar to the one of the derivation of Saint-Venant equations from the Navier-Stokes equations (see [21]).

It is interesting to mention several works about analytical solutions and the comparison between 3D solutions and 2D solutions obtained after applying the Dupuit-Forchheimer approximation: see in particular [26, 36, 25, 22, 23]. For more information about analytical solutions, see [28, 30].

We refer to [17] for the analysis of a model similar to (1.7) in the confined case and [33] for the analysis of a stationary model similar to (1.7),(1.9) (and (1.10)) in the confined case.

Finally, for the identification of hydraulic conductivities, let us mention for instance [16].

1.4 Organization of the paper

In Section 2 we prove Theorem 1.2 and prove Theorem 1.8 in Section 3. In Section 4 we rewrite the models under special assumptions and present some explicit particular stationary solutions.

2 Proof of Theorem 1.2

In this section we give the proof of Theorem 1.2, which is separated in two subsections: the unconfined case and the confined case. In order to simplify the notation, we also denote $(h^\varepsilon, g^\varepsilon)$ by (h, g), and will make some formal computations with (h, g).

2.1 The unconfined case

Step 1: preliminaries
We can rewrite the system (1.2) as

$$(2.1)\qquad \begin{cases} \left.\begin{array}{l} \widetilde{\operatorname{div}}_{\tilde x}\ \tilde v_\alpha^{\tilde x} + \partial_{\tilde z}\tilde v_\alpha^{\tilde z} = 0 \quad \text{in} \quad \tilde\Omega_\alpha^{\tilde t} \\ \tilde v_\alpha = -\tilde\kappa_\alpha(\tilde x, \tilde z)\ \widetilde\nabla(\tilde p + \gamma_\alpha \tilde z) \quad \text{in} \quad \tilde\Omega_\alpha^{\tilde t} \end{array}\right| \quad \text{for} \quad \alpha = f, s \\ \tilde v_f^{\tilde x}\tilde h_{\tilde x} - \tilde v_f^{\tilde z} + \tilde\phi_f(\tilde x, \tilde z)\tilde h_{\tilde t} = 0 \quad \text{on} \quad \left\{(\tilde x, \tilde z) \in \mathbb{R}^N \times \mathbb{R}, \quad \tilde z = \tilde h(\tilde t, \tilde x) < \tilde a(\tilde x)\right\} \\ \tilde v_f^{\tilde x}\tilde g_{\tilde x} - \tilde v_f^{\tilde z} + \tilde\phi_f(\tilde x, \tilde z)\tilde g_{\tilde t} = 0 \quad \text{on} \quad \left\{(\tilde x, \tilde z) \in \mathbb{R}^N \times \mathbb{R}, \quad \tilde z = \tilde g(\tilde t, \tilde x) < \tilde a(\tilde x)\right\} \\ \tilde v_s^{\tilde x}\tilde g_{\tilde x} - \tilde v_s^{\tilde z} + \tilde\phi_s(\tilde x, \tilde z)\tilde g_{\tilde t} = 0 \quad \text{on} \quad \left\{(\tilde x, \tilde z) \in \mathbb{R}^N \times \mathbb{R}, \quad \tilde z = \tilde g(\tilde t, \tilde x) < \tilde a(\tilde x)\right\} \\ \tilde v_s^{\tilde x}\tilde b_{\tilde x} - \tilde v_s^{\tilde z} = 0 \qquad \text{on} \quad \left\{(\tilde x, \tilde z) \in \mathbb{R}^N \times \mathbb{R}, \quad \tilde z = \tilde b(\tilde x) < \tilde g(\tilde t, \tilde x)\right\} \\ \tilde p \quad \text{is continuous on} \quad \Gamma_{\tilde g}^{\tilde t} \\ \tilde v_f \cdot \tilde n \geq 0 \quad \text{on} \quad (\partial\tilde\Omega) \cap (\partial\tilde\Omega_f^{\tilde t}) \\ \tilde v_s \cdot \tilde n \geq 0 \quad \text{on} \quad (\partial\tilde\Omega) \cap (\partial\tilde\Omega_s^{\tilde t}) \cap \left\{\tilde z > \tilde h_1\right\} \end{cases}$$

where the unit vector $\tilde n$ points in the same direction as $\begin{pmatrix} -\widetilde\nabla_{\tilde x}\tilde a(\tilde x) \\ 1 \end{pmatrix} = \begin{pmatrix} -\varepsilon\nabla_x a(x) \\ 1 \end{pmatrix}$
and

$$(2.2)\qquad \begin{cases} \tilde p(\tilde t, \tilde x, \tilde z) = \begin{cases} \gamma_s(\tilde h_1 - \tilde z) & \text{if} \quad \tilde z = \tilde h(\tilde t, \tilde x) = \tilde a(\tilde x) < \tilde h_1 \\ 0 & \text{otherwise} \end{cases} \\ \tilde h = \tilde a \quad \text{if} \quad \tilde a < \tilde h_1 \end{cases}$$

Notice that the equalities on the interfaces $\tilde{z} = \tilde{h}$, $\tilde{z} = \tilde{g}$ and $\tilde{z} = \tilde{b}$, follow from the interpretation of the first equation of (1.2) in the sense of distributions for a function $\tilde{\rho}_\alpha$ which has a discontinuity on those interfaces.

Step 2: rescaling
We set

$$
(2.3) \qquad \begin{cases} \tilde{p}(\tilde{t},\tilde{x},\tilde{z}) = \varepsilon \bar{p}(t,x,z) \\ \tilde{v}_\alpha^{\tilde{x}}(\tilde{t},\tilde{x}) = \varepsilon v_\alpha^x(t,x) \\ \tilde{v}_\alpha^{\tilde{z}}(\tilde{t},\tilde{x}) = \varepsilon^2 v_\alpha^z(t,x). \end{cases}
$$

We get

$$
(2.4) \qquad \begin{cases} \left.\begin{array}{l} \operatorname{div}_x\, v_\alpha^x + \partial_z v_\alpha^z = 0 \quad \text{in} \quad \Omega_\alpha^t \\ -\varepsilon v_\alpha^x = \varepsilon \kappa_\alpha^{xx}(x,z)\nabla_x \bar{p} + \ \kappa_\alpha^{xz}(x,z)\partial_z(\bar{p}+\gamma_\alpha z) \quad \text{in} \quad \Omega_\alpha^t \\ -\varepsilon^2 v_\alpha^z = \varepsilon \kappa_\alpha^{zx}(x,z)\nabla_x \bar{p} + \ \kappa_\alpha^{zz}(x,z)\partial_z(\bar{p}+\gamma_\alpha z) \quad \text{in} \quad \Omega_\alpha^t \end{array}\right| \quad \text{for} \quad \alpha = f,s \\ v_f^x h_x - v_f^z + \phi_f(x,z)h_t = 0 \quad \text{on} \quad \big\{(x,z)\in\mathbb{R}^N\times\mathbb{R}, \quad z = h(t,x) < a(x)\big\} \\ v_f^x g_x - v_f^z + \phi_f(x,z)g_t = 0 \quad \text{on} \quad \big\{(x,z)\in\mathbb{R}^N\times\mathbb{R}, \quad z = g(t,x) < a(x)\big\} \\ v_s^x g_x - v_s^z + \phi_s(x,z)g_t = 0 \quad \text{on} \quad \big\{(x,z)\in\mathbb{R}^N\times\mathbb{R}, \quad z = g(t,x) < a(x)\big\} \\ v_s^x b_x - v_s^z = 0 \qquad \text{on} \quad \big\{(x,z)\in\mathbb{R}^N\times\mathbb{R}, \quad z = b(x) < g(t,x)\big\} \\ \bar{p} \quad \text{is continuous on} \quad \Gamma_g^t \\ -v_f^x\cdot\nabla_x a + v_f^z \geq 0 \quad \text{on} \quad \{z = h(t,x) = a(x) > g(t,x)\} \\ -v_s^x\cdot\nabla_x a + v_s^z \geq 0 \quad \text{on} \quad \{z = a(x) = g(t,x) > h_1\} \end{cases}
$$

with

$$
(2.5) \qquad \begin{cases} \bar{p}(t,x,z) = \begin{cases} \gamma_s \max(0,h_1 - z) & \text{if} \quad z = h(t,x) = a(x) < h_1 \\ 0 & \text{otherwise} \end{cases} \\ h = a \quad \text{if} \quad a < h_1 \end{cases}
$$

This implies in particular that

$$
(2.6) \qquad \left.\begin{cases} \partial_z(\bar{p}+\gamma_\alpha z) & = O(\varepsilon) = -\varepsilon(\kappa_\alpha^{zz}(x,z))^{-1}\left\{\varepsilon v_\alpha^z + \kappa_\alpha^{zx}(x,z)\nabla_x\bar{p}\right\} \\ v_\alpha^x + \bar{\kappa}_\alpha^{xx}(x,z)\nabla_x\bar{p} & = O(\varepsilon) = \varepsilon\kappa_\alpha^{xz}(x,z)(\kappa_\alpha^{zz}(x,z))^{-1}v_\alpha^z \end{cases}\right| \quad \text{in} \quad \Omega_\alpha^t
$$

with

$$\bar{\kappa}_\alpha^{xx}(x,z) := \kappa_\alpha^{xx}(x,z) - (\kappa_\alpha^{zz}(x,z))^{-1}\kappa_\alpha^{xz}(x,z)\kappa_\alpha^{zx}(x,z). \tag{2.7}$$

It is easy to check that the matrix $\bar{\kappa}_\alpha^{xx}(x,z) \in \mathbb{R}^{2\times 2}_{sym}$ is symmetric definite positive because κ_α is symmetric definite positive.

We make the following approximation as ε goes to zero, putting to zero the right hand side of (2.6). This gives

$$\left\{\begin{array}{l} \partial_z(\bar{p}+\gamma_\alpha z) = 0 \\ \\ v_\alpha^x = -\bar{\kappa}_\alpha^{xx}(x,z)\nabla_x \bar{p} \end{array}\right| \quad \text{in} \quad \Omega_\alpha^t \tag{2.8}$$

The second equation of (2.8) gives a kind of effective Darcy's law for the horizontal "velocity" of the fluid. The first equation of (2.8) means that the fluid is vertically at the hydrostatic equilibrium. This implies also that the velocity of the fluid is only horizontal, which is the Dupuit-Forchheimer assumption (see for instance [34], [24]).

Then we can integrate the pressure on the vertical and get

$$\bar{p}(t,x,z) = \left\{\begin{array}{lll} \gamma_s p_0(t,x) + \gamma_f(h(t,x)-z) & & \text{for} \quad g(t,x) < z < h(t,x) \\ \\ \gamma_s p_0(t,x) + \gamma_f(h(t,x)-g(t,x)) + \gamma_s(g(t,x)-z) & & \text{for} \quad b(x) < z < g(t,x) \end{array}\right. \tag{2.9}$$

with

$$p_0(t,x) := \left\{\begin{array}{lll} h_1 - a(x) & & \text{if} \quad h(t,x) = a(x) < h_1 \\ \\ 0 & & \text{otherwise} \end{array}\right.$$

i.e.

$$\gamma_s^{-1}\bar{p}(t,x,z) = \left\{\begin{array}{lll} p_0(t,x) + (1-\varepsilon_0)(h(t,x)-z) & & \text{for} \quad g(t,x) < z < h(t,x) \\ \\ p_0(t,x) + (1-\varepsilon_0)(h(t,x)-g(t,x)) + (g(t,x)-z) & & \text{for} \quad b(x) < z < g(t,x) \end{array}\right.$$

with

$$\varepsilon_0 = \frac{\gamma_s - \gamma_f}{\gamma_s} \in (0,1).$$

This shows in particular that

$$\gamma_s^{-1}\nabla_x \bar{p}(t,\cdot) = \left\{\begin{array}{lll} \nabla_x\left(p_0 + (1-\varepsilon_0)h\right) & & \text{on} \quad \Omega_f^t \\ \\ \nabla_x\left(p_0 + (1-\varepsilon_0)h + \varepsilon_0 g\right) & & \text{on} \quad \Omega_s^t. \end{array}\right. \tag{2.10}$$

Step 3: integration on $[g,h]$
We get

$$\int_g^h dz \ \left(\mathrm{div}_x\ v_f^x(x,z)\right) + \left[v_f^z\right]_{z=g}^{z=h} = 0$$

i.e.

$$\int_g^h dz \ \left(\mathrm{div}_x\ v_f^x(x,z)\right) + \phi_f(x,h)h_t - \phi_f(x,g)g_t + (v_f^x)_{|z=h}h_x - (v_f^x)_{|z=g}g_x = 0.$$

Notice that this equation holds in $\{h < a\}$ where we have used the fourth and the fifth lines of (2.4). On the set $\{h = a > g\}$, we get the same equation if we now interprete the term h_t (which is indeed equal to zero) as a general non-negative quantity $\phi_f(x,h)h_t \geq 0$ (this outflow condition being a straightforward consequence of the line before the last line of (2.4)). With the same convention of interpretation, this can be rewritten as

$$(\Phi_f(x,h) - \Phi_f(x,g))_t + \operatorname{div}_x \left(\int_g^h dz\ v_f^x(x,z) \right) = 0.$$

Using the fact that $\nabla_x \bar{p}$ is independent on z in Ω_f^t, and setting

$$\mathcal{K}_\alpha(x,z) = \gamma_s \int_0^z d\bar{z}\ \bar{\kappa}_\alpha^{xx}(x,\bar{z})$$

we get

$$(\Phi_f(x,h) - \Phi_f(x,g))_t = \operatorname{div}_x \left([\mathcal{K}_f(x,z)]_{z=g}^{z=h} \gamma_s^{-1} \nabla_x \bar{p} \right)$$

i.e.

(2.11)

$$\begin{cases} (\Phi_f(x,h) - \Phi_f(x,g))_t = \operatorname{div}_x \left([\mathcal{K}_f(x,z)]_{z=g}^{z=h} \nabla_x (p_0 + (1-\varepsilon_0)h) \right) & \text{in} \quad \{h < a\} \\ -(\Phi_f(x,g))_t \leq \operatorname{div}_x \left([\mathcal{K}_f(x,z)]_{z=g}^{z=h} \nabla_x (p_0 + (1-\varepsilon_0)h) \right) & \text{in} \quad \{h = a > g\} \end{cases}$$

Step 4: integration on $[b,g]$

We get

$$\int_b^g dz\ \left(\operatorname{div}_x\ v_s^x(x,z)\right) + [v_s^z]_{z=b}^{z=g} = 0$$

i.e.

$$\int_b^g dz\ \left(\operatorname{div}_x\ v_s^x(x,z)\right) + \phi_s(x,g)g_t + (v_s^x)_{|z=g} g_x - (v_s^x)_{|z=b} b_x = 0.$$

Notice that this equation holds in $\{g < a\}$ where we have used the sixth and the seventh lines of (2.4). On the set $\{g = a > h_1\}$, we get the same equation if we now interprete the term g_t (which is indeed equal to zero) as a general non-negative quantity $\phi_s(x,g)g_t \geq 0$ (this outflow condition being a straightforward consequence of the last line of (2.4)). With the same convention of interpretation, this can be rewritten as

$$(\Phi_s(x,g))_t + \operatorname{div}_x \left(\int_b^g dz\ v_s^x(x,z) \right) = 0.$$

Using the fact that $\nabla_x p$ is independent on z in Ω_s^t, we get

$$(\Phi_s(x,g))_t = \operatorname{div}_x \left([\mathcal{K}_s(x,z)]_{z=b}^{z=g} \gamma_s^{-1} \nabla_x \bar{p} \right)$$

i.e.

$$(2.12) \quad \begin{cases} (\Phi_s(x,g))_t = \operatorname{div}_x \left([\mathcal{K}_s(x,z)]_{z=b}^{z=g} \nabla_x (p_0 + (1-\varepsilon_0)h + \varepsilon_0 g) \right) & \text{in} \quad \{g < a\} \\ (\Phi_s(x,g))_t \leq \operatorname{div}_x \left([\mathcal{K}_s(x,z)]_{z=b}^{z=g} \nabla_x (p_0 + (1-\varepsilon_0)h + \varepsilon_0 g) \right) & \text{in} \quad \{g = a > h_1\} \end{cases}$$

Therefore (2.11) and (2.12) imply point a)i) of Theorem 1.2, at least formally if $g = h = a$.

Step 5: Proof of b)i.2) in the evolution case
We assume that at $t = 0$

$$g = a \quad \text{on} \quad \{a < h_1\}$$

and want to show that this is true for all time $t \geq 0$. From the second line of (2.11) we have

$$-(\Phi_f(x,g))_t \leq -\varepsilon_0 \, \mathrm{div}_x \, \left([\mathcal{K}_f(x,z)]_{z=g}^{z=a} \nabla_x a \right) \quad \text{in} \quad \{a < h_1\} \tag{2.13}$$

Integrating on

$$\omega_{h_1} = \{a < h_1\}$$

we get that

$$m(t) := \int_{\omega_{h_1}} (\Phi_f(\cdot\,,a) - \Phi_f(\cdot\,,g)) \geq 0$$

satisfies

$$\frac{dm}{dt} \leq -\varepsilon_0 \int_{\partial\omega_{h_1}} \bar{\nu}^T \cdot [\mathcal{K}_f(x,z)]_{z=g}^{z=a} \nabla_x a \leq 0$$

where the right hand side is non positive because $\bar{\nu} = \frac{\nabla a}{|\nabla a|}$. Therefore $m(t) = 0$ and

$$g = a \quad \text{on} \quad \omega_{h_1}$$

for all time $t \geq 0$. Moreover, we obviously have $h = g = a$ where $g = a$.

Step 6: Proof of b)i.1) in the stationary case
For any $K < h_1$, let us define the set

$$\omega_K = \{a < K\}$$

In the case $g_t = 0$, integrating by parts (2.13) on ω_K, we get

$$0 \leq -\int_{\partial\omega_K} \bar{\nu}^T \cdot [\mathcal{K}_f(x,z)]_{z=g}^{z=a} \cdot \nabla_x a$$

where $\bar{\nu} = \frac{\nabla a}{|\nabla a|}$. This implies that

$$g = a \quad \text{on} \quad \partial\omega_K$$

Because this is true for any $K < h_1$, we conclude that

$$g = a \quad \text{on} \quad \omega_{h_1}$$

2.2 The confined case

Step 1: proof of a)ii)
The procedure is exactly the same as in the unconfined case, except that h is independent on the time t, and that the renormalized pressure $p_0(x)$ is replaced by

$$p(t,x) = \begin{cases} \max(0, h_1 - h(x)) > 0 & \text{for} \quad x \in \mathbb{R}^N \backslash \omega \\ \\ \text{unknown} & \text{for} \quad x \in \omega = \{h < a\} \end{cases}$$

Therefore we have

(2.14)
$$\begin{cases} -(\Phi_f(x,g))_t &= \operatorname{div}_x \ \Big([\mathcal{K}_f(x,z)]_{z=g}^{z=h} \nabla_x (p + (1-\varepsilon_0)h) \Big) \qquad \text{in} \quad \{h<a\} = \omega \\ \\ -(\Phi_f(x,g))_t &\le \operatorname{div}_x \ \Big([\mathcal{K}_f(x,z)]_{z=g}^{z=h} \nabla_x (p + (1-\varepsilon_0)h) \Big) \qquad \text{in} \quad \{g<h=a\} \subset \mathbb{R}^N \backslash \omega \\ \\ (\Phi_s(x,g))_t &= \operatorname{div}_x \ \left([\mathcal{K}_s(x,z)]_{z=b}^{z=g} \nabla_x (p + (1-\varepsilon_0)h + \varepsilon_0 g) \right) \qquad \text{in} \quad \{g<a\} \supset \omega \\ \\ 0 &\le \operatorname{div}_x \ \left([\mathcal{K}_s(x,z)]_{z=b}^{z=g} \nabla_x (p + (1-\varepsilon_0)h + \varepsilon_0 g) \right) \qquad \text{in} \quad \{g=a\} \subset \mathbb{R}^N \backslash \omega. \end{cases}$$

This implies in particular that p solves the following equation

$$\begin{aligned} &(\Phi_s(x,g) - \Phi_f(x,g))_t \\ &= \operatorname{div}_x \ \Big(\Big([\mathcal{K}_s(x,z)]_{z=b}^{z=g} + [\mathcal{K}_f(x,z)]_{z=g}^{z=h} \Big) \nabla_x (p_0 + (1-\varepsilon_0)h) + \varepsilon_0 \, [\mathcal{K}_s(x,z)]_{z=b}^{z=g} \nabla_x g) \Big) \quad \text{in} \quad \omega. \end{aligned}$$

This implies the result ii) of Theorem 1.2 in the confined case.

Step 2: proof of b)ii.2) in the evolution case

We recall that

$$(2.15) \qquad -(\Phi_f(\cdot\,,g))_t \le \operatorname{div}_x \ \Big([\mathcal{K}_f(x,z)]_{z=g}^{z=h} \nabla_x (p + (1-\varepsilon_0)h) \Big) \quad \text{on} \quad [0,+\infty) \times \mathbb{R}^N$$

Integrating (by parts) this inequality on $\bar{\omega} = \mathbb{R}^N \backslash \omega$, we get (using $g \le a$) that

$$m(t) := \int_{\bar{\omega}} (\Phi_f(\cdot\,,a) - \Phi_f(\cdot\,,g)) \ge 0$$

satifies

$$\frac{dm}{dt} \le -\varepsilon_0 \int_{\partial\bar{\omega}} \bar{\nu}^T \cdot [\mathcal{K}_f(x,z)]_{z=g}^{z=a} \nabla_x a \le 0$$

where $\bar{\nu} = -\nu$ is the outward unit normal to $\bar{\omega}$, and where we have used condition (1.11). We then conclude as in the unconfined case (Step 5).

Step 3: proof of b)ii.1) in the stationary case

For any $K \in \mathbb{R}$, let us define the set

$$\bar{\omega}_K = \bar{\omega} \cap \{a < K\}$$

In the case $g_t = 0$, integrating by parts (2.15) on $\bar{\omega}_K$, we get

$$0 \le -\int_{\partial\bar{\omega}_K} \bar{\nu}^T \cdot [\mathcal{K}_f(x,z)]_{z=g}^{z=a} \cdot \nabla_x a$$

where

$$\bar{\nu} = \begin{cases} -\nu & \text{if} \quad x \in \partial\bar{\omega} \\ \\ \frac{\nabla_x a}{|\nabla_x a|} & \text{if} \quad x \in \partial\bar{\omega}_K \backslash \partial\bar{\omega} \end{cases}$$

This implies that

$$g = a \quad \text{on} \quad \partial\bar{\omega}_K$$

Because this is true for any $K \in \mathbb{R}$, we conclude that

$$g = a \quad \text{on} \quad \bar{\omega}.$$

3 Proof of the general Ghyben-Herzberg relation

Proof of Theorem 1.8
i) (unconfined case)
From Theorem 1.2 b) i.1), we have $g = a = h$ on $\{a < h_1\}$. Moreover we have $p = 0$ on $\{a \geq h_1\} \supset \{g < a\}$, and then we deduce in particular from the fourth line of (1.7) that

$$(3.1) \qquad \begin{cases} \mathrm{div}_x \ \left([\mathcal{K}_s(x,z)]_{z=b}^{z=g} \nabla_x((1-\varepsilon_0)h + \varepsilon_0 g)\right) = 0 & \text{on } \ D := \{b < g < a\} \\ \\ h = g = h_1 & \text{on } \ \Gamma_a := \{g = a\} \cap \partial D \\ \\ g = b & \text{on } \ \Gamma_b := \{g = b\}, \end{cases}$$

The second line of (3.1) follows from the fact that

$$(3.2) \qquad \{a > h_1\} \subset \{g < a\}$$

because we assume $g \leq h_1$ in i) of Theorem 1.8. Indeed, recall that $D \subset \{a \geq h_1\}$. Therefore $\Gamma_a \subset \{a \geq h_1\}$. Moreover $\Gamma_a \cap \{a > h_1\} = \emptyset$, because of (3.2). Therefore $\Gamma_a \subset \{a = h_1\}$ and then $g = a = h_1 = h$ on Γ_a, which shows the second line of (3.1).

Let

$$\Psi := (1-\varepsilon_0)h + \varepsilon_0 g - h_1.$$

Using (3.1), we have

$$(3.3) \qquad \begin{cases} \mathrm{div}_x \ \left([\mathcal{K}_s(x,z)]_{z=b}^{z=g} \nabla_x \Psi\right) = 0 & \text{on } \ D \\ \\ \Psi = 0 & \text{on } \ \Gamma_a \\ \\ g = b & \text{on } \ \Gamma_b. \end{cases}$$

Multiplying the first equation in (3.3) by Ψ and integrating over D, we get

$$0 = \int_D \Psi \mathrm{div}_x \left([\mathcal{K}_s(x,z)]_{z=b}^{z=g} \nabla_x \Psi\right) = -\int_D (\nabla_x \Psi)^T [\mathcal{K}_s(x,z)]_{z=b}^{z=g} (\nabla_x \Psi) \leq 0,$$

where the boundary terms vanish because of the two last lines of (3.3). This implies that $\nabla_x \Psi = 0$ on D and therefore $\Psi = cst$ locally. Therefore $\Psi = 0$ on each connected component of D whose boundary intersects Γ_a.
ii) (confined case)
From Theorem 1.2 b) ii.1), we have $g = a$ on $\{h = a\}$. From the fourth line of (1.7), we deduce in particular that

$$(3.4) \qquad \begin{cases} \mathrm{div}_x \ \left([\mathcal{K}_s(x,z)]_{z=b}^{z=g} \nabla_x(p + (1-\varepsilon_0)h + \varepsilon_0 g)\right) = 0 & \text{on } \ D := \{b < g < a\} \\ \\ h = g = a & \text{on } \ \Gamma_a := \{g = a\} \cap \partial D \\ \\ g = b & \text{on } \ \Gamma_b := \{g = b\}, \end{cases}$$

Notice that the second line of (3.4) is automatic because $g \leq h \leq a$. Let

$$\Psi := p + (1-\varepsilon_0)h + \varepsilon_0 g - h_1.$$

Using (3.4) and (1.9), we get again (3.3) and conclude as in the unconfined case i). This completes the proof of the theorem. $\square$

4 Special assumptions and particular solutions

In this section, in order to simplify, we assume that

$$\mathcal{K}_\alpha(x,z) = z \cdot Id. \tag{4.1}$$

and

$$\phi_\alpha \equiv 1$$

We will present some explicit stationary solutions.

4.1 Unconfined aquifer

A particular solution of (1.7) is a solution of

$$\begin{cases} b \le g \le h \le a & \text{on} \quad [0,+\infty) \times \mathbb{R}^N \\ (h-g)_t = \operatorname{div}_x \ ((h-g)\nabla_x((1-\varepsilon_0)h)) & \text{on} \quad \{h < a\} \\ g_t = \operatorname{div}_x \ ((g-b)\nabla_x((1-\varepsilon_0)h + \varepsilon_0 g)) & \text{on} \quad \{g < a\} \\ g = a = h & \text{on} \quad \{a \le h_1\} \\ h < a & \text{on} \quad \{a > h_1\} \end{cases} \tag{4.2}$$

whose a particular stationary solution satisfying the Ghyben-Herzberg condition

$$g = \max(b, h_1 - (1-\varepsilon_0)\bar{h}) \quad \text{on} \quad \{g < a\}$$

with $\bar{h} = h - g \ge 0$, $\bar{g} = g - b \ge 0$, solves

$$\begin{cases} h = \max(\bar{h} + b, h_1 + \varepsilon_0 \bar{h}) & \text{on} \quad \{h < a\} \\ 0 = \operatorname{div}_x \ \left(\bar{h}\nabla_x \left\{\max(\bar{h} + b, h_1 + \varepsilon_0\bar{h})\right\}\right) & \text{on} \quad \{h < a\} = \{a > h_1\} \\ \bar{h} = 0 & \text{on} \quad \{h = a\} = \{a \le h_1\} \end{cases} \tag{4.3}$$

An example of a particular stationary solution

We recover a classical Ghyben-Herzberg solution in a special case. Let us consider the case $b = 0 < h_1 = a(0)$, $N = 1$ and then $x = x_1 \in \mathbb{R}$. We also assume that

$$\begin{cases} g(x) < h(x) < a(x) & \text{for} \quad x < 0 \\ g(x) = h(x) = a(x) & \text{for} \quad x \ge 0. \end{cases}$$

We set

$$h_0 = \frac{h_1}{1-\varepsilon_0}.$$

Then for any $\ell > 0$, there exists an explicit solution of (4.3) given by

$$\bar{h}(x) = \begin{cases} h_0 \sqrt{\dfrac{-x}{\ell}} & \text{for} \quad -\ell \leq x < 0 \\[2ex] h_0 \sqrt{1 - \varepsilon_0 \left(\dfrac{x+\ell}{\ell} \right)} & \text{for} \quad x \leq -\ell. \end{cases}$$

This corresponds to

$$g(x) = \begin{cases} h_1 - (1-\varepsilon_0) h_0 \sqrt{\dfrac{-x}{\ell}} & \text{for} \quad -\ell \leq x < 0 \\[2ex] 0 & \text{for} \quad x \leq -\ell, \end{cases}$$

$$h(x) = \begin{cases} h_1 + \varepsilon_0 h_0 \sqrt{\dfrac{-x}{\ell}} & \text{for} \quad -\ell \leq x < 0 \\[2ex] h_0 \sqrt{1 - \varepsilon_0 \left(\dfrac{x+\ell}{\ell} \right)} & \text{for} \quad x \leq -\ell. \end{cases}$$

Thi solution is represented on Figure 3.

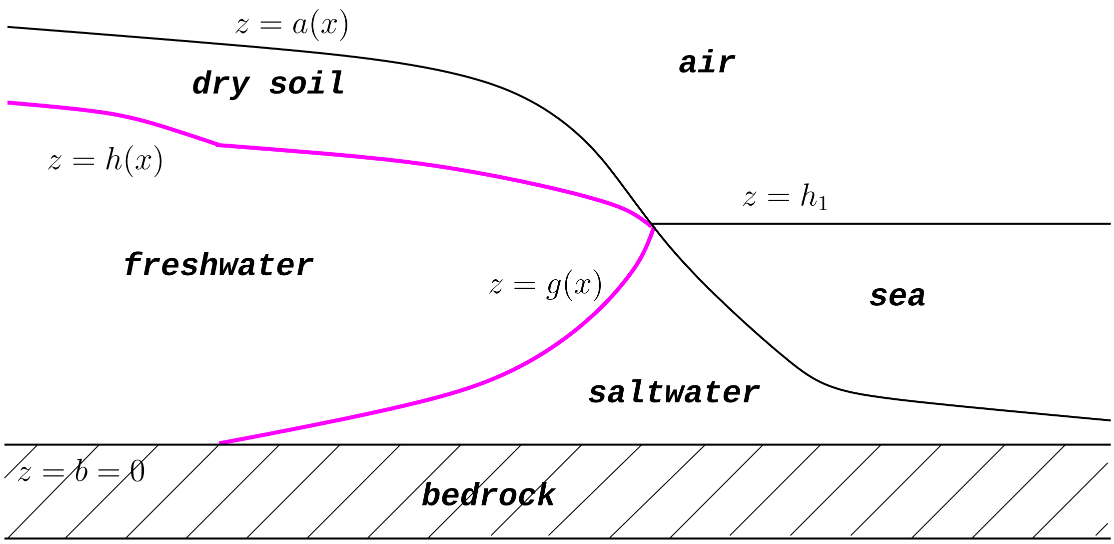

Figure 3: A classical stationary solution in the unconfined case

Remark 4.1 *Notice that, because of the square root shape of the free boundary at the point $x = 0$, and the infinite velocity of the freshwater at that point (because this field is divergence free), the assumption (2.3) of small and essentially horizontal velocity is no longer valid at that point.*

4.2 Confined aquifer

We recall that

$$\omega = \left\{ x \in \mathbb{R}^N, \quad h(x) < a(x) \right\}$$

Then a particular solution of (1.7),(1.9) is a solution of

$$(4.4) \qquad \begin{cases} b \le g \le h \le a & \text{on} \quad [0,+\infty) \times \mathbb{R}^N \\ g_t = \mathrm{div}_x \ \left((g-b)\nabla_x (p + (1-\varepsilon_0)h + \varepsilon_0 g)\right) & \text{on} \quad \omega \\ g = a = h & \text{on} \quad \mathbb{R}^N \backslash \omega \end{cases}$$

with p solution of

$$(4.5) \qquad \begin{cases} 0 = \mathrm{div}_x \ \left((h-b)\nabla_x (p + (1-\varepsilon_0)h) + \varepsilon_0 (g-b)\nabla_x g)\right) & \text{on} \quad \omega \\ p(t,x) = p_1(x) := \max(0, h_1 - a(x)) > 0 & \text{on} \quad \partial\omega. \end{cases}$$

Proposition 4.2 (horizontal confinement)
We assume that $h \equiv h_0 \in (0, h_1)$ on ω and $b \equiv 0$. Then the solution g of (4.4)-(4.5) satisfies for all time $t > 0$

$$(4.6) \qquad \begin{cases} 0 \le g \le h_0 & \textit{on} \quad \omega \\ g = h_0 & \textit{on} \quad \partial\omega \\ g_t = \varepsilon_0 \ \mathit{div}_x \ \left(g\left(1 - \dfrac{g}{h_0}\right)\nabla_x g \right) + \varepsilon_0 \ \mathit{div}_x \ (g \nabla_x w) & \textit{on} \quad \omega \end{cases}$$

with w solution of

$$(4.7) \qquad \begin{cases} \Delta w = 0 & \textit{on} \quad \omega \\ w = \dfrac{h_0}{2} & \textit{on} \quad \partial\omega. \end{cases}$$

Remark 4.3 *Notice that if $g = 0$ in a region of ω far enough from the boundary $\partial\omega$, then the velocity of the freshwater is proportional to $\nabla_x w$, and the flux $\nabla_x w$ can then be assumed to be a given quantity. This can allow to solve equation (4.7).*

Remark 4.4 *The third equation of (4.6) appears to be an approximation of the equation considered in [34], in the limit case where the the gradient of the solution is small.*

Proof of Proposition 4.2
We simply set

$$w(t,x) = \varepsilon_0^{-1}(p(t,x) - (h_1 - h_0)) + \frac{1}{2h_0} g^2(t,x).$$

The rest of the proof is straightforward.

4.2.1 A stationary solution

Proposition 4.5 (A stationary solution in the confined case)
We work under the assumptions of Proposition 4.2. We consider a solution w of

$$\left\{\begin{array}{lll} \Delta w = 0 & \quad on & \omega \\ \\ w = \dfrac{h_0}{2} & \quad on & \partial\omega \end{array}\right.$$

and we assume the following Ghyben-Herzberg relation

$$g - \frac{g^2}{2h_0} + w = h_0 \quad on \quad \{g > 0\}\,. \tag{4.8}$$

Then any solution of (4.6)-(4.7)-(4.8) satisfies

$$\left\{\begin{array}{l} g_t = 0 \\ \\ \{g > 0\} = \{w < h_0\}\,. \end{array}\right. \tag{4.9}$$

Proof of Proposition 4.5
Straightforward.

An example of a particular stationary solution
We will find a solution in dimension 1 (in the permanent regime). Let us consider the case $b = 0 < h_0 = a(0)$, $N = 1$ and then $x = x_1 \in \mathbb{R}$. We assume that

$$\omega = \{x < 0\}\,.$$

Under the assumptions of Proposition 4.5, for any $\ell > 0$, there exists an explicit solution given by

$$w(x) = \frac{h_0}{2}\left(1 - \frac{x}{\ell}\right)$$

and

$$\left\{\begin{array}{lll} g - \dfrac{g^2}{2h_0} + w = h_0 & \quad \text{for} & -\ell < x < 0 \\ \\ g = 0 & \quad \text{for} & x \le -\ell. \end{array}\right.$$

This solution is represented on Figure 4.

Acknowledgments
The authors would like to thank A. Al Bitar and R. Ababou for enlighting discussions and for indications of the literature on the subject. A discussion with R. Ababou was useful to correct a mistake in a previous version of the manuscript. The authors also would like to thank A. Fino for helpful discussions.

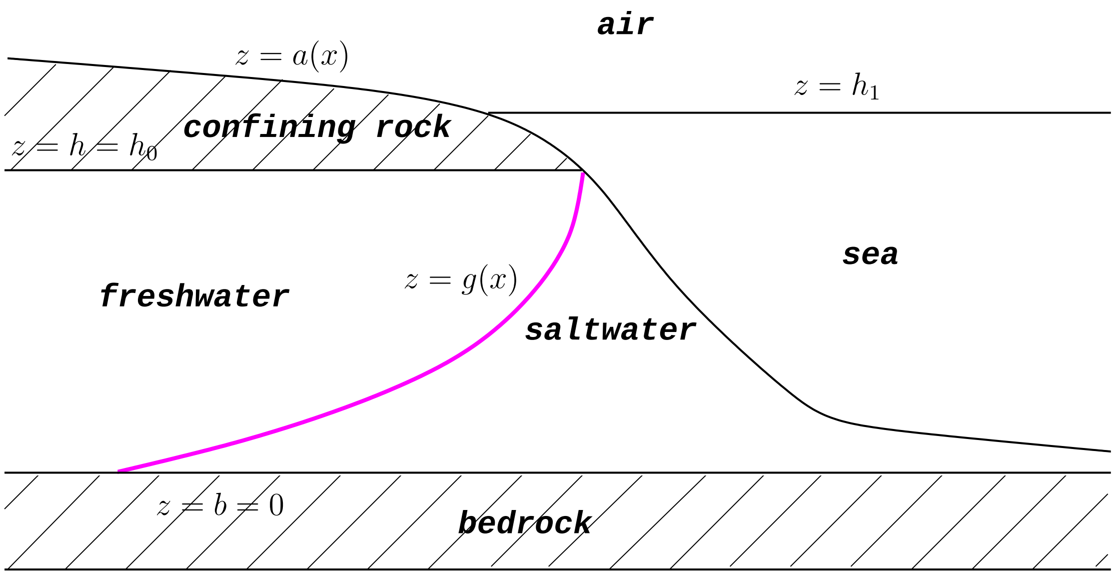

Figure 4: A stationary solution in the confined case

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